Problem 1

Suppose L is lower triangular invertible and I want to solve

$$LX = B. \tag{1}$$

where B is a n by n matrix. The straightforward substitution method is $O\left(n^3\right)$. Design a Strassen-style fast method that solves the linear system without forming L^{-1} first (to subsequently evaluate $X = L^{-1}B$).

Solution. Conformally block partition $LX = B$ into:

$$\begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

We can rewrite it as

$$\begin{aligned} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} &= \begin{bmatrix} L_{11}^{-1} & \\ -L_{22}^{-1} L_{21} L_{11}^{-1} & L_{22}^{-1} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} L_{11}^{-1} B_{11} & L_{11}^{-1} B_{21} \\ -L_{22}^{-1} L_{21} L_{11}^{-1} B_{11} + L_{22}^{-1} B_{21} & -L_{22}^{-1} L_{21} L_{11}^{-1} B_{12} + L_{22}^{-1} B_{22} \end{bmatrix} \\ &= \begin{bmatrix} L_{11}^{-1} B_{11} & L_{11}^{-1} B_{21} \\ L_{22}^{-1}(B_{21} - L_{21} L_{11}^{-1} B_{11}) & L_{22}^{-1}(B_{22} - L_{21} L_{11}^{-1} B_{12}) \end{bmatrix} \end{aligned}$$

We follow the following procedure to solve $LX = B$:

1. Recursively solve $L_{11}X_{11} = B_{11}$ for X_{11} and $L_{11}X_{21} = B_{21}$ for X_{21}.
2. Make a call to Strassen matrix-matrix multiply to evaluate $L_{21}Z$ and $L_{21}W$.
3. Perform the additions $B_{21} - L_{21}X_{11}$ and $B_{22} - L_{21}L_{11}^{-1}X_{21}$
4. Recursively solve $L_{22}X_{21} = B_{21} - L_{21}L_{11}^{-1}B_{11}$ and $L_{22}X_{22} = B_{22} - L_{21}L_{11}^{-1}B_{12}$

Let ω denote the exponent of matrix-matrix multiplication (in the case of Strassen, $\omega \approx 2.8074$). We have the following recursion to solve

$$f(n) \leq 4f(n/2) + Cn^{\omega}$$

Since $\log_2 4 = 2 < \omega$, invoking the master equation overall leads to a complexity of $O\left(n^{\omega}\right)$. □

Problem 2

Let $\epsilon > 0$ and let k be any integer. Prove that

$$\lim_{n \to \infty} \frac{n(\log n)^k}{n^{1+\epsilon}} = 0.$$

Solution. Substituting $n = 2^t$ yields

$$\lim_{n \to \infty} \frac{n(\log n)^k}{n^{1+\epsilon}} = (\log 2)^k \lim_{t \to \infty} \frac{t^k}{(2^{\epsilon})^t}$$

Since $2^{\epsilon} > 1$ for every $\epsilon > 0$, the above limit should evaluate to 0. □