Problem 1

Let $\varphi \in \{\mathbb{C} \setminus 0\}$. Consider the matrix

$$Z_\varphi = \begin{bmatrix} & & \cdots & & \varphi \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \vdots \\ & & & 1 & \end{bmatrix} \in \mathbb{C}^{N \times N}$$

Determine an explicit expression for the eigendecomposition of $Z_\varphi = V \Lambda V^{-1}$ and discuss how one can efficiently compute the products Vx and $V^{-1}x$.

Problem 2

Consider the problem of finding the best L^2-approximation of a function through a linear combination of cosines, i.e.,

$$\inf_{c \in \mathbb{R}^n} \left(f(\theta) - \sum_{k=1}^{n} c_k \cos(k\theta) \right)$$

The entries of the normal equation $Ac = b$ for this problem take on the form

$$a_{kl} = \int_a^b \cos(k\theta)\cos(l\theta) d\theta, \qquad b_k = \int_a^b \cos(k\theta) f(\theta) d\theta.$$

Make use of trigonometric identies to show that $A \in \mathbb{R}^{n \times n}$ is the sum of a Toeplitz and a Hankel matrix. Furthermore, verify that such matrices have low displacement rank for the displacement operator $Y_{\phi,\delta}A - AY_{\gamma,\sigma}$, where

$$Y_{\phi,\delta} = \begin{bmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \delta \end{bmatrix} + \begin{bmatrix} \phi & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & \end{bmatrix}$$

What is the displacement rank?

Problem 3

Let A be a square non-singular matrix and suppose that

$$\Omega \begin{bmatrix} A & G^* \\ F & B \end{bmatrix} - \begin{bmatrix} A & G^* \\ F & B \end{bmatrix} \Lambda = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}^*$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. By inserting Gauss transforms:

$$H_1 = \begin{bmatrix} I & \\ -FA^{-1} & I \end{bmatrix}, \qquad H_2 = \begin{bmatrix} I & -A^{-1}G^* \\ & I \end{bmatrix}$$

at appropriate locations in the above equation, find the corresponding displacement rank equation for the Schur complement $B - FA^{-1}G^*$.

Problem 4

Using displacement rank theory and the block matrix

$$\begin{bmatrix} I & T_2 \\ T_1 & 0 \end{bmatrix}$$

to describe a procedure to multiply two Toeplitz matrices $T_1 T_2$ in $\Theta\left(n^2\right)$ flops.

Problem 4

Determine an explicit expression for the inverse of

$$A = \begin{bmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ & a_2 & 1 & & & \\ & & a_3 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & a_{n-1} & 1 \end{bmatrix}$$

What can be said about the ranks of the lower off-diagonal blocks?