Problem 1

Let $\varphi \in \{\mathbb{C} \setminus 0\}$. Consider the matrix

$$Z_\varphi = \begin{bmatrix} & & \cdots & & \varphi \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \vdots \\ & & & 1 & \end{bmatrix} \in \mathbb{C}^{N\times N}$$

Determine an explicit expression for the eigendecomposition of $Z_\varphi = V\Lambda V^{-1}$ and discuss how one can efficiently compute the products Vx and $V^{-1}x$.

Solution. Observe that Z_ϕ is a companion matrix with characteristic equation $\lambda^n - \phi = 0$. Let $\bar{\omega}_n = e^{2\pi i/n}$. If $\varphi^{1/n}$ is the principal n-th root of $\varphi \in \{\mathbb{C} \setminus 0\}$, the eigenvalues and corresponding left-eigenmatrix entries of Z_ϕ can be expressed as $\lambda_k = \varphi^{1/n}\bar{\omega}_n^{k-1}$ and $w_{kl} = \frac{1}{\sqrt{n}}\varphi^{(l-1)/n}\bar{\omega}_n^{(k-1)(l-1)}$ for $k, l = 1, n$, respectively. We thus observe that the left-eigenmatrix is a column-scaled inverse DFT matrix, i.e.,

$$V^{-1} = W = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \cdots & \bar{\omega}^{n-1} \\ 1 & \bar{\omega}^2 & \bar{\omega}^3 & \cdots & \bar{\omega}^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{\omega}^{n-1} & \bar{\omega}^{(n-1)2} & \cdots & \bar{\omega}^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \varphi^{1/n} & & & \\ & & \varphi^{2/n} & & \\ & & & \ddots & \\ & & & & \varphi^{(n-1)/n} \end{bmatrix} = F_n^* D_\varphi^{-1}$$

We therefore have the eigendecomposition

$$Z_\varphi = (D_\varphi F_n)(\varphi^{1/n}\Omega)(D_\varphi F_n)^{-1},$$

where $\Omega = \operatorname{diag}(1, \bar{\omega}_n, \ldots, \bar{\omega}_n^{n-1})$ Computing Vx and $V^{-1}x$ can be done fast in $O(n \log n)$ as one simply needs to apply a FFT in combination with a diagonal scaling. ☐

Problem 2

Consider the problem of finding the best L^2-approximation of a function through a linear combination of cosines, i.e.,

$$\inf_{c\in\mathbb{R}^n} \left(f(\theta) - \sum_{k=1}^{n} c_k \cos(k\theta) \right)$$

The entries of the normal equation $Ac = b$ for this problem take on the form

$$a_{kl} = \int_a^b \cos(k\theta)\cos(l\theta)d\theta, \qquad b_k = \int_a^b \cos(k\theta)f(\theta)d\theta.$$

Make use of trigonometric identies to show that $A \in \mathbb{R}^{n\times n}$ is the sum of a Toeplitz and a Hankel matrix. Furthermore, verify that such matrices have low displacement rank for the displacement operator $Y_{\phi,\delta}A - AY_{\gamma,\sigma}$, where

$$Y_{\phi,\delta} = \begin{bmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \delta \end{bmatrix} + \begin{bmatrix} \phi & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & \end{bmatrix}$$

What is the displacement rank?

Solution. The matrix A is the sum of a Toeplitz and a Hankel matrix, because

$$\begin{aligned} a_{kl} &= \int_a^b \cos(k\theta)\cos(l\theta)d\theta \\ &= \int_a^b \tfrac{1}{2}\left(\cos(k\theta)\cos(l\theta)+\sin(k\theta)\sin(l\theta)\right)d\theta + \int_a^b \tfrac{1}{2}\left(\cos(k\theta)\cos(l\theta)-\sin(k\theta)\sin(l\theta)\right)d\theta \\ &= \int_a^b \tfrac{1}{2}\cos((k-l)\theta)d\theta + \int_a^b \tfrac{1}{2}\cos((k+l)\theta)d\theta \\ &= t(k-l)+h(k+l). \end{aligned}$$

Evaluating the displacement operator we get something of the form:

$$Y_{\phi,\delta}A - AY_{\gamma,\sigma} = \begin{bmatrix} * & * & * & \cdots & * \\ * & & & & * \\ * & & & & * \\ \vdots & & & & \vdots \\ * & * & * & \cdots & * \end{bmatrix}$$

The right-hand-side is a rank 4 matrix. Hence, the displacement rank is 4. □

Problem 3

Let A be a square non-singular matrix and suppose that

$$\Omega \begin{bmatrix} A & G^* \\ F & B \end{bmatrix} - \begin{bmatrix} A & G^* \\ F & B \end{bmatrix} \Lambda = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}^*$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. By inserting Gauss transforms:

$$H_1 = \begin{bmatrix} I & \\ -FA^{-1} & I \end{bmatrix}, \qquad H_2 = \begin{bmatrix} I & -A^{-1}G^* \\ & I \end{bmatrix}$$

at appropriate locations in the above equation, find the corresponding displacement rank equation for the Schur complement $B - FA^{-1}G^*$.

Solution. We perform the manipulation:

$$H_1\Omega H_1^{-1}H_1 \begin{bmatrix} A & G^* \\ F & B \end{bmatrix} H_2 - H_1 \begin{bmatrix} A & G^* \\ F & B \end{bmatrix} H_2H_2^{-1}\Lambda H_2 = H_1 \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}^* H_2$$

The reduces to:

$$\Omega \begin{bmatrix} A & \\ & B - FA^{-1}G^* \end{bmatrix} - \begin{bmatrix} A & \\ & B - FA^{-1}G^* \end{bmatrix} \Lambda = \begin{bmatrix} R_1 \\ R_2 - FA^{-1}R_1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 - G(A^*)^{-1}S_1 \end{bmatrix}^*$$

Hence $B - FA^{-1}G^*$ satisfies the displacement equation

$$\tilde{\Omega}(B - FA^{-1}G^*) - (B - FA^{-1}G^*)\tilde{\Lambda} = \tilde{R}\tilde{S}^*,$$

where:

$$\tilde{\Omega} = \begin{bmatrix} \omega_2 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix}, \qquad \tilde{\Lambda} = \begin{bmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \qquad \tilde{R} = R_2 - FA^{-1}R_1, \qquad \tilde{S} = S_2 - G(A^*)^{-1}S_1.$$

□

Problem 4

Using displacement rank theory and the block matrix

$$\begin{bmatrix} I & T_2 \\ T_1 & 0 \end{bmatrix}$$

to describe a procedure to multiply two Toeplitz matrices T_1T_2 in $\Theta\left(n^2\right)$ flops.

Solution. The first observation we make is that computing T_1T_2 is equivalent to performing one step of block-LU, since

$$\begin{bmatrix} I & T_2 \\ T_1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ T_1 & I \end{bmatrix} \begin{bmatrix} I & T_2 \\ 0 & -T_1T_2 \end{bmatrix}$$

The second observation we make is that $\begin{bmatrix} I & T_2 \\ T_1 & 0 \end{bmatrix}$ is a matrix of low displacement rank, since

$$\begin{aligned}
\begin{bmatrix} I & T_2 \\ T_1 & 0 \end{bmatrix} - Z_{2n,\downarrow} \begin{bmatrix} I & T_2 \\ T_1 & 0 \end{bmatrix} Z_{2n,\downarrow}^\top &= \left[\begin{array}{ccccc|ccccc}
1 & & & & & \tau_0 & \tau_{-1} & \tau_{-2} & \cdots & \tau_{-n+1} \\
 & 0 & & & & \tau_1 & & & & \\
 & & 0 & & & \tau_2 & & & & \\
 & & & \ddots & & \vdots & & & & \\
 & & & & 0 & \tau_{n-1} & & & & \\
\hline
t_0 & t_{-1} & t_{-2} & \cdots & t_{-n+1} & 1 & & & & \\
t_1 & & & & & & & & & \\
t_2 & & & & & & & & & \\
\vdots & & & & & & & & & \\
t_{n-1} & & & & & & & & &
\end{array}\right] \\
&= \left[\begin{array}{cccc}
1 & \tau_0 & 0 & 1 \\
0 & \tau_1 & 0 & 0 \\
0 & \tau_2 & 0 & 0 \\
\vdots & \vdots & 0 & 0 \\
0 & \tau_{n-1} & 0 & 0 \\
\hline
t_0 & 1 & 1 & 0 \\
t_1 & 0 & 0 & 0 \\
t_2 & 0 & 0 & 0 \\
\vdots & \vdots & 0 & 0 \\
t_{n-1} & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & t_{-1} & t_{-2} & \cdots & t_{-n+1} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \tau_{-1} & \tau_{-2} & \cdots & \tau_{-n+1}
\end{array}\right] \\
&= \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}^\top
\end{aligned}$$

The third observation we make is that the displacement operators $Z_{2n,\downarrow}$ and $Z_{2n,\downarrow}^\top$ are lower and upper triangular, respectively. Thus one can obtain a compact representation ($O\left(n\right)$ numbers) of the product of two Toeplitz matrices by running the Schur algorithm. The time complexity of this operation is $O\left(n^2\right)$.

□

Problem 4

Determine an explicit expression for the inverse of

$$A = \begin{bmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ & a_2 & 1 & & & \\ & & a_3 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & a_{n-1} & 1 \end{bmatrix}$$

What can be said about the ranks of the lower off-diagonal blocks?

Solution. We use von Neumann series $(I - X)^{-1} = I + X + X^2 + X^3 + \dots$ to evaluate inverse of A. Write

$$A = I + Z_\downarrow \mathrm{diag}(\boldsymbol{a}), \qquad \boldsymbol{a} = (a_1, a_2, \dots, a_n)$$

Hence, taking note of the nilpotency of $Z_\downarrow$, we may write

$$\begin{aligned} A^{-1} &= (I - (-Z_\downarrow \mathrm{diag}(\boldsymbol{a})))^{-1} \\ &= I - Z_\downarrow \mathrm{diag}(\boldsymbol{a}) + (Z_\downarrow \mathrm{diag}(\boldsymbol{a}))^2 - (Z_\downarrow \mathrm{diag}(\boldsymbol{a}))^3 + \dots + (-1)^{n-1}(Z_\downarrow \mathrm{diag}(\boldsymbol{a}))^{n-1} \\ &= \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} - \begin{bmatrix} & & & & \\ a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_{n-1} & \end{bmatrix} + \begin{bmatrix} & & & \\ & & & \\ a_1a_2 & & & \\ & a_2a_3 & & \\ & & \ddots & \\ & & & a_{n-2}a_{n-1} & & \end{bmatrix} \\ &\quad - \begin{bmatrix} & & & \\ & & & \\ & & & \\ a_1a_2a_3 & & & \\ & \ddots & & \\ & & a_{n-3}a_{n-2}a_{n-1} & & & \end{bmatrix} + \dots + (-1)^{n-1} \begin{bmatrix} & & \\ & & \\ & & \\ & & \\ a_1a_2\cdots a_{n-1} & & \end{bmatrix} \end{aligned}$$

Hence,

$$A_{ij}^{-1} = \begin{cases} 0 & i < j \\ 1 & i = j \\ (-1)^{i-j} \prod_{k=j}^{i-1} a_k & i > j \end{cases}$$

As proven in class, the ranks of the off-diagonal (i.e., Hankel) blocks of a matrix is invariant under inversion. We recognize this property also in the inverse of A. The upper off-diagonal blocks of the inverse are of zero rank, whereas the lower off-diagonal blocks are of unit rank, since

$$A^{-1}(i+1:n, 1:i) = \begin{bmatrix} a_i \\ -a_ia_{i+1} \\ \vdots \\ (-1)^{n-i}a_ia_{i+1}\cdots a_{n-1} \end{bmatrix} \begin{bmatrix} (-1)^{n-i-1}a_{i-1} & \cdots & a_1 & -1 \end{bmatrix}.$$

$\square$