A tensor-based approach to solving systems of multivariate polynomials

Nithin Govindarajan

with Raphaël Widdershoven, Shiv Chandrasekaran (UCSB), and Lieven De Lathauwer

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KU LEUVEN

Overview

Motivation: noisy overdetermined polynomial systems

Polynomial root solving: from an eigenvalue to a tensor decomposition problem

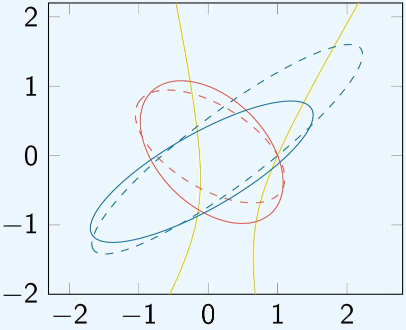
Faster Macaulay null space computations

Summary

Noisy overdetermined systems: looking for approximative roots

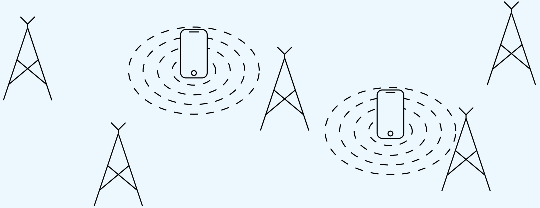
$$\begin{cases} -3 - x - 2y + 4x^2 + 6xy + 7y^2 = 0 \\ -2 - x + y + 3x^2 - 7xy + 5y^2 = 0 \\ 1 + 7x + y - 8x^2 + 3xy + y^2 = 0 \end{cases}$$

- $N = 2$ unknowns
- $S = 3$ equations $\rightarrow$ *overdetermined*
- Degree $d = 2$



Adding noise to the red and blue equations *destroys* the single exact root at $(1, 0)$

A practical application: "blind" multi-source localization



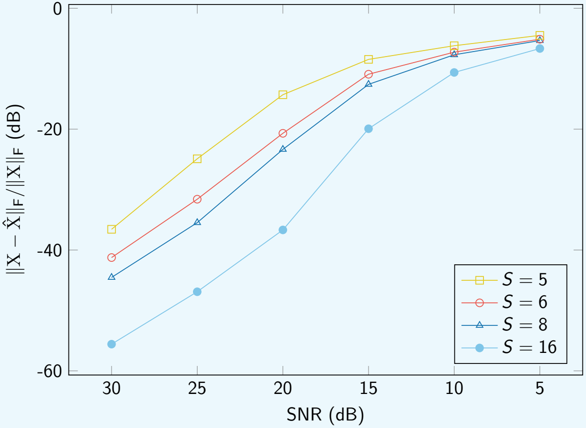
Friis transmission equation (before conversion into a polynomial expression):

$$P_i^r = \frac{A_i^r A_1^t}{\lambda^2} \frac{P_1^t}{(x_i^r - x_1^t)^2 + (y_i^r - y_1^t)^2} + \frac{A_i^r A_2^t}{\lambda^2} \frac{P_2^t}{(x_i^r - x_2^t)^2 + (y_i^r - y_2^t)^2}, \quad i = 1, \ldots, S.$$

Noisy measured quantities! Unknown Given

$\longrightarrow$ for $S \geq 5$, positions of transmitters can be *retrieved* up to permutation ambiguity!

Similar to least-squares: adding more equations (i.e., antennas) yield better estimates



Median relative error of estimated transmitter positions over 200 experiments
(Widdershoven, Govindarajan, et al. 2023)

-50 dB error $\approx$ 5 digits of precision

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Algebraic methods: "classical" vs. recent numerical (multi)-linear algebra approaches

Find all (projective) roots of the system of the multivariate polynomials:

$$\Sigma : \quad \left\{ \begin{array}{rcl} p_1 & = & p_1(x_1, x_2, \ldots, x_N) \\ & \vdots & \\ p_S & = & p_S(x_1, x_2, \ldots, x_N) \end{array} \right. , \qquad S \geq N, \qquad \deg(p_s) = d_s.$$

Auzinger, Stetter, Lazard, ...	Batselier, Dreesen,...	Vanderstukken, De Lathauwer,...
Numerical Polynomial Algebra (NPA) →	Numerical Polynomial Linear Algebra (NPLA) →	Numerical Polynomial Multi-Linear Algebra (NPMLA)
Features: - Gröbner basis construction - eigenvalue problem	Features: - Macaulay null space construction - Generalized eigenvalue problem	Features: - Macaulay null space construction - tensor decomposition problem

DISCLAIMER: The above is a very selected overview and only shows "ancestors" of our own work. It is by no means a summary of all the contributions done on this topic.

The Macaulay-based method for polynomial root solving

The rows of the Macaulay matrix $\mathrm{M}(d)$ span the set

$$\mathcal{M}_d := \left\{ \sum_{s=1}^{S} g_s \cdot p_s : \quad \deg(g_s) = d - d_s \right\}.$$

For example, $M(3)$ for the system in slide 3:

$$\begin{array}{c}
p_1(x,y) \\ p_2(x,y) \\ p_3(x,y) \\ xp_1(x,y) \\ xp_2(x,y) \\ xp_3(x,y) \\ yp_1(x,y) \\ yp_2(x,y) \\ yp_3(x,y)
\end{array}
\left[\begin{array}{c|cc|ccc|cccc}
-3 & -1 & -2 & 4 & 6 & 7 & 0 & 0 & 0 & 0 \\
-2 & -1 & 1 & 3 & -7 & 5 & 0 & 0 & 0 & 0 \\
1 & 7 & 1 & -8 & 3 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & -3 & 0 & -1 & -2 & 0 & 4 & 6 & 7 & 0 \\
0 & -2 & 0 & -1 & 1 & 0 & 3 & -7 & 5 & 0 \\
0 & 1 & 0 & 7 & 1 & 0 & -8 & 3 & 1 & 0 \\
\hline
0 & 0 & -3 & 0 & -1 & -2 & 0 & 4 & 6 & 7 \\
0 & 0 & -2 & 0 & -1 & 1 & 0 & 3 & -7 & 5 \\
0 & 0 & 1 & 0 & 7 & 1 & 0 & -8 & 3 & 1
\end{array}\right]
\left[\begin{array}{c}
1 \\ \hline x \\ y \\ \hline x^2 \\ xy \\ y^2 \\ \hline x^3 \\ x^2y \\ xy^2 \\ y^3
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0
\end{array}\right]$$

If $d \geq d^*$ (degree of regularity), $\dim \operatorname{null} \mathrm{M}(d) =$ no. of projective roots of the system.

The Macaulay-based method for polynomial root solving

The rows of the Macaulay matrix $\mathrm{M}(d)$ span the set

$$\mathcal{M}_d := \left\{ \sum_{s=1}^{S} g_s \cdot p_s : \quad \deg(g_s) = d - d_s \right\}.$$

For example, $M(3)$ for the system in slide 3:

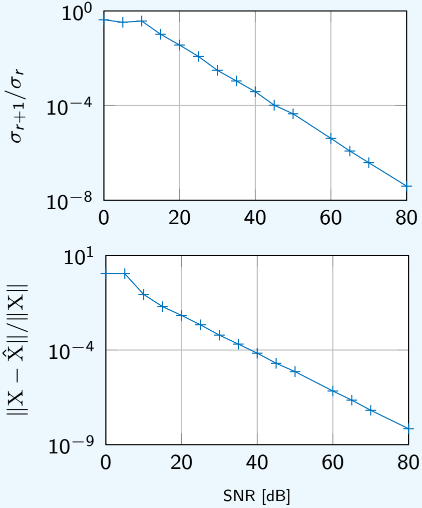
$$\begin{array}{r}
t^3 p_1(x/t,y/t) \\
t^3 p_2(x/t,y/t) \\
t^3 p_3(x/t,y/t) \\
t^2 x p_1(x/t,y/t) \\
t^2 x p_2(x/t,y/t) \\
t^2 x p_3(x/t,y/t) \\
t^2 y p_1(x/t,y/t) \\
t^2 y p_2(x/t,y/t) \\
t^2 y p_3(x/t,y/t)
\end{array}
\left[\begin{array}{c|cc|ccc|cccc}
-3 & -1 & -2 & 4 & 6 & 7 & 0 & 0 & 0 & 0 \\
-2 & -1 & 1 & 3 & -7 & 5 & 0 & 0 & 0 & 0 \\
1 & 7 & 1 & -8 & 3 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & -3 & 0 & -1 & -2 & 0 & 4 & 6 & 7 & 0 \\
0 & -2 & 0 & -1 & 1 & 0 & 3 & -7 & 5 & 0 \\
0 & 1 & 0 & 7 & 1 & 0 & -8 & 3 & 1 & 0 \\
\hline
0 & 0 & -3 & 0 & -1 & -2 & 0 & 4 & 6 & 7 \\
0 & 0 & -2 & 0 & -1 & 1 & 0 & 3 & -7 & 5 \\
0 & 0 & 1 & 0 & 7 & 1 & 0 & -8 & 3 & 1
\end{array}\right]
\left[\begin{array}{c}
t^3 \\ \hline t^2 x \\ t^2 y \\ \hline t x^2 \\ t x y \\ t y^2 \\ \hline x^3 \\ x^2 y \\ x y^2 \\ y^3
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0
\end{array}\right]$$

If $d \geq d^*$ (degree of regularity), $\dim \operatorname{null} \mathrm{M}(d) =$ no. of projective roots of the system.

Our system from slide 3: Macaulay method is suitable in the noisy setting

Add noise to the nonzero coefficients:

$$M(4) = \left[\begin{array}{ccccccccccccccc}
-3 & -1 & 4 & & & -2 & 6 & & & 7 & & & & & \\
-2 & -1 & 3 & & & 1 & 7 & & & 5 & & & & & \\
1 & 7 & -8 & & & 1 & 3 & & & 1 & & & & & \\
& -3 & -1 & 4 & & & -2 & 6 & & & 7 & & & & \\
& -2 & -1 & 3 & & & 1 & 7 & & & 5 & & & & \\
& 1 & 7 & -8 & & & 1 & 3 & & & 1 & & & & \\
& & -3 & -1 & 4 & & & -2 & 6 & & & 7 & & & \\
& & -2 & -1 & 3 & & & 1 & 7 & & & 5 & & & \\
& & 1 & 7 & -8 & & & 1 & 3 & & & 1 & & & \\
& & & & & -3 & -1 & 4 & & -2 & 6 & & 7 & & \\
& & & & & -2 & -1 & 3 & & 1 & 7 & & 5 & & \\
& & & & & 1 & 7 & -8 & & 1 & 3 & & 1 & & \\
& & & & & & -3 & -1 & 4 & & -2 & 6 & & 7 & \\
& & & & & & -2 & -1 & 3 & & 1 & 7 & & 5 & \\
& & & & & & 1 & 7 & -8 & & 1 & 3 & & 1 & \\
& & & & & & & & & -3 & -1 & 4 & -2 & 6 & 7 \\
& & & & & & & & & -2 & -1 & 3 & 1 & 7 & 5 \\
& & & & & & & & & 1 & 7 & -8 & 1 & 3 & 1
\end{array}\right]$$



Note: $M(4)$ is expressed in lex ordering this time!

The matrix view: recovering the roots from a generalized eigenvalue problem

- Let $\mathrm{S}_t, \mathrm{S}_{x_1}, \mathrm{S}_{x_2}, \ldots, \mathrm{S}_{x_N}$ denote appropriate "row selection" matrices.
- Construct $\mathrm{G}_t = \mathrm{S}_t \mathrm{N}$ and $\mathrm{G}_{x_i} = \mathrm{S}_{x_i} \mathrm{N}$ for $i = 1, \ldots, N$ with col $\mathrm{N} = \text{null}\, \mathrm{M}(d)$
- Solve the generalized eigenvalue decomposition (GEVD) problem:

$$\left(\alpha_t \mathrm{G}_t + \alpha_{x_1} \mathrm{G}_{x_1} + \ldots + \alpha_{x_N} \mathrm{G}_{x_N}\right) \boldsymbol{a} = \lambda \left(\beta_t \mathrm{G}_t + \beta_{x_1} \mathrm{G}_{x_1} + \ldots + \beta_{x_N} \mathrm{G}_{x_N}\right) \boldsymbol{a}$$

- For $i = 1, \ldots, R$ (number of roots), we have the eigenvalues:

$$\lambda_i = \frac{\alpha_t t^{(i)} + \alpha_{x_1} x_1^{(i)} + \ldots + \alpha_{x_N} x_N^{(i)}}{\beta_t t^{(i)} + \beta_{x_1} x_1^{(i)} + \ldots + \beta_{x_N} x_N^{(i)}}$$

- Eigenvectors *reveal* root location, since

$$\boldsymbol{v}_i = \left(\alpha_t \mathrm{G}_t + \alpha_{x_1} \mathrm{G}_{x_1} + \ldots + \alpha_{x_N} \mathrm{G}_{x_N}\right) \boldsymbol{a}_i$$

are multivariate *Vandermonde* vectors evaluated at the system roots!

Typically $\alpha_{x_1}, \ldots, \alpha_{x_n}$ is set to zero.

Matrix pencils can be badly conditioned when eigenvalues coalesce...

Theorem (See e.g., Golub and Van Loan 2012))

Define $\mathsf{sep}(\mathrm{T}_{11}, \mathrm{T}_{22}) := \min \|\mathrm{T}_{11}\mathrm{X} - \mathrm{X}\mathrm{T}_{22}\|_F \,/\, \|\mathrm{X}\|_F$ *and let* $\mathrm{Q}^*\mathrm{A}\mathrm{Q} = \begin{bmatrix} \mathrm{T}_{11} & \mathrm{T}_{12} \\ 0 & \mathrm{T}_{22} \end{bmatrix}$ *with* $\mathrm{Q} = \begin{bmatrix} \mathrm{Q}_1 & \mathrm{Q}_2 \end{bmatrix}$. *Then for a "sufficiently small"* E, *there exists a* $\tilde{\mathrm{Q}}_1$ *with*

$$\mathsf{dist}(\mathsf{col}\ \mathrm{Q}_1, \mathsf{col}\ \tilde{\mathrm{Q}}_1) \lesssim \frac{1}{\mathsf{sep}(\mathrm{T}_{11}, \mathrm{T}_{22})}$$

such that $\tilde{\mathrm{Q}}_1$ *is an invariant subspace for* $\tilde{\mathrm{A}} = \mathrm{A} + \mathrm{E}$.

A bad choice of α's and β's can bring eigenvalues arbitrarily close!

The above is a compressed statement carrying the essence of Corollary 7.2.5 in (Golub and Van Loan 2012)

GESD principle: why limit to just one pencil, if you can exploit multiple?

GESD algorithm: using multiple pencils, recursively split eigenspaces corresponding to well-separated eigenvalue clusters (Evert, Vandecappelle, et al. 2022).

Simultaneous diagonalization:

There exists an invertible $\mathrm{A} \in \mathbb{C}^{R\times R}$ that simultaneously diagonalizes

$$\begin{aligned} \mathrm{G}_t \mathrm{A} &= \mathrm{V}\,\mathrm{diag}(t^{(1)}, \ldots, t^{(R)}), \\ \mathrm{G}_{x_1} \mathrm{A} &= \mathrm{V}\,\mathrm{diag}(x_1^{(1)}, \ldots, x_1^{(R)}), \\ &\vdots \\ \mathrm{G}_{x_N} \mathrm{A} &= \mathrm{V}\,\mathrm{diag}(x_N^{(1)}, \ldots, x_N^{(R)}), \end{aligned}$$

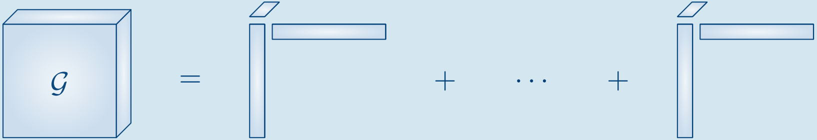
with V being multivariate Vandermonde matrix *evaluated* at the roots.

Reformulation of the root recovery as a tensor decomposition problem

Theorem (Vanderstukken and De Lathauwer 2021)

Let $\mathcal{G}$ *have frontal slices* $\mathrm{G}_t, \mathrm{G}_{x_1}, \ldots, \mathrm{G}_{x_2}$, *and assume* Σ *has only simple roots. If* $\mathcal{G}$ *is constructed from* $\operatorname{null} \mathrm{M}(d)$ *with* $d \geq d^* + 1$, *then* $\mathcal{G}$ *has the essentially unique CPD*

$$\mathcal{G} = [\![\mathrm{V}, \mathrm{A}^{-1}, \mathrm{X}]\!] \,, \quad X = \begin{bmatrix} t^{(1)} & t^{(2)} & \cdots & t^{(R)} \\ x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(R)} \\ \vdots & \vdots & & \vdots \\ x_N^{(1)} & x_N^{(2)} & \cdots & x_N^{(R)} \end{bmatrix}.$$

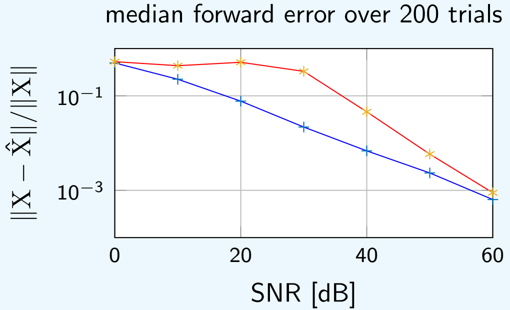


If the polynomial system has roots of multiplicity greater than one, the theorem can be generalized with the introduction of block-term decompositions (Vanderstukken, Kürschner, et al. 2021)

Observed numerical benefits of the tensor approach for noisy overdetermined systems

Take $N = 10$ *noisy* copies of the square system:

$$\Sigma : \begin{cases} f_1(x_1, x_2) = x_1^3 + x_2^3 - 9x_1^2 x_2 + 20x_1 x_2 - 3x_1 - 20 = 0 \\ f_2(x_1, x_2) = x_1^2 + 4x_2^2 - x_1 x_2 - 80 = 0 \end{cases}$$



The tensor-based method that relies on simultaneous diagonalization is better capable of recovering roots in noisy conditions than a pure matrix-based method which relies solely on GEVD (Vanderstukken and De Lathauwer 2021).

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Null space computation is the major computational bottleneck in *many* algorithms!

$$
\begin{tikzcd}
\Sigma \arrow[r, "\text{Homogenize}"] \arrow[d] & \Sigma_h \arrow[r, dashed, "\text{Determine roots}"] & \boxed{\left\{\left(t^{(r)}, x_1^{(r)}, \ldots, x_N^{(r)}\right)\right\}_{r=1}^{R}} \\
\mathrm{M}(d) \arrow[r, "\text{Compute null space basis}"] & \mathrm{N}(d) \arrow[r, "\text{Solve joint-GEVD}"] & \begin{matrix} \mathrm{diag}(t^{(1)}, \ldots, t^{(R)}), \\ \mathrm{diag}(x_1^{(1)}, \ldots, x_1^{(R)}), \\ \vdots \\ \mathrm{diag}(x_N^{(1)}, \ldots, x_N^{(R)}) \end{matrix} \arrow[u]
\end{tikzcd}
$$

Exploit the *Toeplitz* structures in Macaulay matrices?

Bivariate systems: Macaulay matrix is almost Toeplitz block-(block-)Toeplitz

$$M(4) = \begin{array}{cc}
 & \begin{array}{ccccc|cccc|ccc|cc|c} 1 & x & x^2 & x^3 & x^4 & y & xy & x^2y & x^3y & y^2 & xy^2 & x^2y^2 & y^3 & xy^3 & y^4 \end{array} \\
\begin{array}{c} p_1 \\ p_2 \\ xp_1 \\ xp_2 \\ x^2p_1 \\ x^2p_2 \\ \hline yp_1 \\ yp_2 \\ xyp_1 \\ xyp_2 \\ \hline y^2p_1 \\ y^2p_2 \end{array} &
\left[\begin{array}{ccccc|cccc|ccc|cc|c}
1 & 6 & 4 & & & 2 & 5 & & & 3 & & & & & \\
9 & 1 & 3 & & & 8 & 7 & & & 2 & & & & & \\
 & 1 & 6 & 4 & & & 2 & 5 & & & 3 & & & & \\
 & 9 & 1 & 3 & & & 8 & 7 & & & 2 & & & & \\
 & & 1 & 6 & 4 & & & 2 & 5 & & & 3 & & & \\
 & & 9 & 1 & 3 & & & 8 & 7 & & & 2 & & & \\
\hline
 & & & & & 1 & 6 & 4 & & 2 & 5 & & 3 & & \\
 & & & & & 9 & 1 & 3 & & 8 & 7 & & 2 & & \\
 & & & & & & 1 & 6 & 4 & & 2 & 5 & & 3 & \\
 & & & & & & 9 & 1 & 3 & & 8 & 7 & & 2 & \\
\hline
 & & & & & & & & & 1 & 6 & 4 & 2 & 5 & 3 \\
 & & & & & & & & & 9 & 1 & 3 & 8 & 7 & 2
\end{array}\right]
\end{array}$$

The Macaulay matrix for the general bivariate case

Let $\Delta d := d - d_\Sigma$. Then,

$$\mathrm{M}(d) := \begin{bmatrix} \mathrm{M}_{0,0} & \mathrm{M}_{1,0} & \cdots & \mathrm{M}_{d_\Sigma,0} & & & \\ & \mathrm{M}_{0,1} & \mathrm{M}_{1,1} & \cdots & \mathrm{M}_{d_\Sigma,1} & & \\ & & \ddots & \ddots & & \ddots & \\ & & & \mathrm{M}_{0,\Delta d} & \mathrm{M}_{1,\Delta d} & \cdots & \mathrm{M}_{d_\Sigma,\Delta d} \end{bmatrix} \in \mathbb{C}^{\frac{S}{2}(\Delta d+1)(\Delta d+2)\times\frac{1}{2}(d+1)(d+2)},$$

with

$$\mathrm{M}_{i,j} := \begin{bmatrix} \boldsymbol{c}_{0i} & \boldsymbol{c}_{1i} & \cdots & \boldsymbol{c}_{(d_\Sigma-i)i} & & & \\ & \boldsymbol{c}_{0i} & \boldsymbol{c}_{1i} & \cdots & \boldsymbol{c}_{(d_\Sigma-i)i} & & \\ & & \ddots & \ddots & & \ddots & \\ & & & \boldsymbol{c}_{0i} & \boldsymbol{c}_{1i} & \cdots & \boldsymbol{c}_{(d_\Sigma-i)i} \end{bmatrix} \in \mathbb{C}^{S(\Delta d+1-j)\times(d+1-i-j)}.$$

The key observation that shall allow for a faster algorithm

Consider the displacement operator

$$\mathscr{D}\{\mathrm{M}(d)\} = \begin{bmatrix} \mathrm{Z}_{d+1,1} \otimes \mathrm{I}_S & & \\ & \ddots & \\ & & \mathrm{Z}_{1,1} \otimes \mathrm{I}_S \end{bmatrix} \mathrm{M}(d) - \mathrm{M}(d) \begin{bmatrix} \mathrm{Z}_{d+1,\varphi_{d+1}} & & \\ & \ddots & \\ & & \mathrm{Z}_{1,\varphi_1} \end{bmatrix}.$$

$\mathrm{M}(d)$ has relative "low" displacement rank

Dimensions of $\mathrm{M}(d) \in \mathbb{C}^{\frac{S}{2}(\Delta d+1)(\Delta d+2) \times \frac{1}{2}(d+1)(d+2)}$ grow *quadratically* w.r.t. d, but

$$\operatorname{rank} \mathscr{D}\{\mathrm{M}(d)\} \leq S(\Delta d + 1) = S\,(d + 1 - d_\Sigma)\,.$$

grows only *linearly* with d.

Here $\mathrm{Z}_{p,\varphi} := \begin{bmatrix} & & \varphi \\ 1 & & \\ & \ddots & \\ & 1 & \end{bmatrix} \in \mathbb{C}^{p \times p}$.

Overview of the fast algorithm

$$
\begin{tikzcd}
\mathrm{M}(d) \arrow[r, dashed, "\text{Determine null space}"] \arrow[d, "\substack{\textbf{step 1}: \\ \text{convert into Cauchy-like} \\ \text{matrix using FFTs}}"'] & \boxed{\mathrm{N}(d) = \Psi \hat{\mathrm{N}}(d)} \\
\hat{\mathrm{M}}(d) = \Phi \mathrm{M}(d) \Psi \arrow[r, "\substack{\textbf{step 2}: \\ \text{compute a rank-revealing} \\ \text{LU factorization of } \hat{\mathrm{M}}(d)}"] & \hat{\mathrm{N}}(d) \arrow[u]
\end{tikzcd}
$$

Both steps can be done *fast*!
(Govindarajan, Widdershoven, et al. 2023)

Rank-revealing LU factorization of $\hat{\mathrm{M}}(d)$ (Miranian and Gu 2003)

Let $r(d) := \operatorname{rank} \mathrm{M}(d)$. Compute a rank-revealing LU (RRLU) factorization

$$\begin{aligned}
\Pi_1 \hat{\mathrm{M}}(d) \Pi_2 &= \begin{bmatrix} \mathrm{I}_{r(d)} & \\ \hat{\mathrm{M}}_{21}\hat{\mathrm{M}}_{11}^{-1} & \mathrm{I}_{d_\Sigma^2} \end{bmatrix} \begin{bmatrix} \hat{\mathrm{M}}_{11} & \\ & \hat{\mathrm{M}}_{22} - \hat{\mathrm{M}}_{21}\hat{\mathrm{M}}_{11}^{-1}\hat{\mathrm{M}}_{12} \end{bmatrix} \begin{bmatrix} \mathrm{I}_{r(d)} & \hat{\mathrm{M}}_{11}^{-1}\hat{\mathrm{M}}_{12} \\ & \mathrm{I}_{d_\Sigma^2} \end{bmatrix} \\
&\approx \begin{bmatrix} \hat{\mathrm{M}}_{11} \\ \hat{\mathrm{M}}_{21} \end{bmatrix} \begin{bmatrix} \mathrm{I}_{r(d)} & \hat{\mathrm{M}}_{11}^{-1}\hat{\mathrm{M}}_{12} \end{bmatrix}
\end{aligned}$$

Expression for the null space $\mathrm{N}(d)$

$$\mathrm{N}(d) = \Psi \Pi_2 \begin{bmatrix} \tilde{\mathrm{N}} \\ \mathrm{I}_{d_\Sigma^2} \end{bmatrix}, \qquad \tilde{\mathrm{N}} := -\hat{\mathrm{M}}_{11}^{-1}\hat{\mathrm{M}}_{12}.$$

The classical Schur algorithm: making things work for us

- Run Schur algorithm on Cauchy-like matrices, which satisfy the displacement relation $\mathrm{diag}(\boldsymbol{\mu})\mathrm{C} - \mathrm{C}\mathrm{diag}(\boldsymbol{\nu}) = \mathrm{R}\mathrm{S}^*$ (Heinig 1995)
- Replace total pivoting with approximate total pivoting (Gu 1998)
- Let $j_{\max}$ denote the column with largest 2-norm in $\mathrm{R}\mathrm{S}^*$. Then,

$$\max_{1\le i\le n} |c_{ij_{\max}}| \ge \frac{1}{K\sqrt{n}} \max_{1\le i,j\le n} |c_{ij}|, \quad K := \max_{1\le i,j,\imath,\jmath\le n} |\mu_i - \nu_j| \,/\, |\mu_\imath - \nu_\jmath|.$$

- Keeping R orthogonal during the Gaussian elimination process allows for fast pivot selection.
- Use clever updating strategies to keep cost low.

RESULT: from $\mathcal{O}(d^6)$ to $\mathcal{O}(d^5)$ (flop count)

Algorithm stability: error grows linearly with problem size

Median error $\epsilon := \|\mathrm{M}(d)\mathrm{Q}\|_2 \,/\, \|\mathrm{M}(d)\|_2 \geq \sigma_{r(d)+1}/\sigma_1$, with $\mathrm{Q} \in \mathbb{C}^{n(d)\times d_\Sigma^2}$ an orthonormal basis for col $\mathrm{N}(d)$, over 100 runs for randomly generated *square* systems.

	d_Σ				
	2	4	8	16	32
SVD on $\mathrm{M}(d)$	2.23e-16	3.75e-16	5.70e-16	7.94e-16	9.51e-16
GECP on $\mathrm{M}(d)$	1.40e-16	3.11e-16	8.33e-16	1.02e-14	1.40e-13
GECP on $\mathscr{C}$	4.35e-16	1.51e-15	1.35e-14	1.72e-13	2.81e-12
GEAP on $\mathscr{C}$	4.21e-16	3.63e-15	3.88e-14	3.19e-13	4.48e-12

Sources of error:

- switching to LU instead of an SVD
- working with the compact Cauchy representation $\mathscr{C}$
- switching to approximate pivoting $\leftarrow$ Surprisingly not so bad!

Extending the method

- Generalizations to Chebyshev systems possible; Toeplitz $\rightarrow$ Toeplitz-plus-Hankel ☺
- Displacement rank theory does not generalize nicely to higher dimensions with diminishing returns $\mathcal{O}(d^{3N})$ to $\mathcal{O}(d^{3N-1})$ ☹
- Assume additional (block) sparsity of Macaulay matrix to make breakthrough with n-dimensional systems?

$$p_i(x,y) = \sum_{k=0}^{d_\Sigma - r} a_{ikr} x^k y^r + \sum_{r\in\mathscr{S}} \sum_{k=0}^{d_\Sigma - r} a_{ikr} x^k y^r, \quad \mathscr{S} \subset \{1,\ldots,d_\Sigma\}, \quad |\mathscr{S}| = \rho \ll d_\Sigma$$

Overview

What we have discussed in this talk

- Solving polynomial systems in the *noisy* overdetermined setting.
- Benefits of taking on a "*tensor*" view towards polynomial root solving.
- Progress and challenges towards (asymptotically) *faster* Macaulay null space algorithms.

References I

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